THREE-PARAMETER LIE GROUPS ADJACENT TO THE GALILEAN AND EUCLIDEAN GROUPS

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Stratification of the space of the structural constants of three-dimensional Lie algebras into the equivalence classes according to the criterion of their isomorphism over the field of real members is performed, and this makes it possible to enumerate the local Lie groups close (in some sense) to the Galilean and Euclidean groups and becoming identical with them at the "limit".

This constitutes the simplest of the problems dealing with the boundedness of the set of the Lie groups "refining" the Lie groups describing the already known physical objects. (The description of a physical object in terms of a group is understood here to conform to the physical treatment of the Klein's Erlangen program [1]). The theory of deformation and contraction of rings and Lie algebras which was receiving attention in the past few years [2–5] is concerned with similar problems.

1. Statement of the problem. Let R denote a real Euclidean tensor space $c_{jk}{}^i$ $(i, j, k \leqslant n)$ and $\Gamma \subset R$ be one of the irreducible manifolds of the structural constants of the Lie groups G which can be separated from R by means of the Jacobi conditions

$$c_{ab}^{e} c_{ec}^{f} + c_{bc}^{e} c_{ea}^{f} + c_{ca}^{e} c_{eb}^{f} = 0, \qquad c_{ec}^{f} = -c_{ce}^{f} \tag{1.1}$$

The space of the structural constants of the groups G is defined as the union of all Γ. The local isomorphism of G is an equivalence relation ([6], p. 18), hence it controls the decomposition of the manifolds Γ into pair-wise nonintersecting sets H. Thus, one-to-one correspondence arises between the sets H and the nonisomorphic local Lie groups (and their algebras).

Let A and B denote the sets of the space R. Here and in the following $\bar{A}$ denotes the closure of A in the real topology for the Euclidean spaces([6], pp. 67, 68) and $\bar{A}_B = \bar{A} \cap B$ is the relative closure of A on B, the latter regarded as a subspace of R. Let also H' and H'' be the different sets belonging either to the same manifold, or to different manifolds Γ. We shall say that the local group G' corresponding to the set H' is adjacent to the local group G'' corresponding to the set H'', if

$$\bar{H}' \cap H'' \neq \Lambda \tag{1.2}$$

where Λ is an empty set.

Obviously, if G' is adjacent to G'', a continuous sequence of groups $G(\lambda)$ locally isomorphic to G' when $0 < \lambda \leqslant \lambda_0$ exists, which converges to the group $G(0)$ locally isomorphic to G'' when $\lambda \to +0$.

The purpose of this paper is to obtain an algebraic description of the partitioning of the manifolds Γ for three-parameter algebras and to enumerate all local groups adjacent to each of the groups G, and to the Galilean and Euclidean groups in particular.

2. The method of solution and the result. Let AG be the algebra of infinitesimal operators of the group G and $X_1, \ldots, X_n$ its basis.

The real linear group $GL(n, R)$ whose basis

$$X_i' = \alpha_i^j X_j \quad (i, j = 1, \ldots, n), \qquad \det(\alpha_i^j) \neq 0$$

is preselected from the basis of AG, consists of two connected sheets corresponding to the motions $(\det(\alpha_i^j) > 0)$ and reflections $(\det(\alpha_i^j) < 0)$ and induces the representation T in the space R. Transformations of the group T

$$c_{ij}^{\delta'} = c_{k_1 k_2}^{\gamma} \alpha_i^{k_1} \alpha_j^{k_2} \alpha_{\gamma}^{*\delta}, \quad \alpha_{\nu}^{\gamma} \alpha_{\gamma}^{*\varepsilon} = \delta_{\nu}^{\varepsilon} \quad (\delta_{\nu}^{\varepsilon} \text{ — Kronecker tensor}) \tag{2.1}$$

conserve the relations (1.1) transforming each of the manifolds Γ into itself. Transformations of the groups induced by T and Γ exhaust all the transformations conserving the local isomorphism of the groups G. Consequently they realize the equivalence relation according to which the decomposition of the manifolds Γ is performed. From this it follows that the sets H are orbits of T and of the groups induced by T on its invariant manifolds.

Let $\Gamma = \Gamma^0 \supset \Gamma^1 \supset \ldots \supset \Gamma^l$ be such a finite sequence of invariant manifolds of the group T, that all invariant manifolds of this group containing Γ^l are represented on it and set the group T be transitive on the manifold Γ^l provided that the latter cannot be reduced to a set of isolated points. Then each orbit of T will either represent the connectivity component of one of the sets

$$\Gamma^0 \setminus \Gamma^1, \quad \Gamma^1 \setminus \Gamma^2, \ldots, \Gamma^{l-1} \setminus \Gamma^l, \quad \Gamma^l \tag{2.2}$$

or it will be the union of two components of the same set, the latter case arising when a reflection exists in the group $GL(n, R)$ generating a homeomorphism of one of these components into the other.

Thus, the solution of the problem formulated above requires that: (1) invariant manifolds of T must be found; (2) its orbits must be computed and (3) the adjacency diagram in the sense of (1.2) must be constructed for G. When the problem is solved in this order as shown in Sects. 3–5, we obtain the following result.

We introduce the following notation:

$$c_{23}^{\ 1} = u, \quad c_{13}^{\ 2} = v, \quad c_{12}^{\ 3} = w \tag{2.3}$$

$$f_1 = c_{12}^{\ 2} + c_{13}^{\ 3}, \quad \theta_1 = c_{12}^{\ 2} - c_{13}^{\ 3}, \quad \chi_1 = 2uf_1 - \theta_3 f_2 + \theta_2 f_3$$

$$f_2 = c_{13}^{\ 1} + c_{23}^{\ 2}, \quad \theta_2 = c_{13}^{\ 1} - c_{23}^{\ 2}, \quad \chi_2 = \theta_3 f_1 - 2wf_2 - \theta_1 f_3$$

$$f_3 = c_{12}^{\ 1} - c_{23}^{\ 3}, \quad \theta_3 = c_{12}^{\ 1} + c_{23}^{\ 3}, \quad \chi_3 = \theta_2 f_1 + \theta_1 f_2 - 2vf_3$$

$$\varphi = \theta_1\theta_2\theta_3 + 4uvw + u\theta_1^2 - v\theta_3^2 + w\theta_2^2$$

$$\psi_1 = \theta_1^2 + 4vw, \quad \psi_2 = \theta_2^2 + 4uv, \quad \psi_3 = \theta_3^2 - 4uw$$

$$\psi_4 = \theta_1\theta_3 + 2w\theta_2, \quad \psi_5 = \theta_1\theta_2 - 2v\theta_3, \quad \psi_6 = \theta_2\theta_3 + 2u\theta_1$$

$$\Omega = \psi_i / f_i^2 \quad \text{for } f_i \neq 0 \quad (i = 1, 2, 3)$$

Symbol $\dim H$ denotes the dimension of the orbit coinciding with the rank of the matrix (3.5) of the infinitesimal operators of the algebra AT associated with the group T computed at the points belonging to this orbit and m is the ordinal number of G according to Scheffers ([7], see also [8], p. 167) classification.

We thus have two irreducible manifolds Γ_1 and Γ_2 of the structural constants and two series of groups G depending on whether the expression $f_1^2 + f_2^2 + f_3^2$ does, or does not vanish.

Groups G belonging to the finite series $(f_1^2 + f_2^2 + f_3^2 = 0)$

G^1 Lobachevskii group (group of motions of the Lobachevskii plane) $\dim H = 6$; $m = 1$

$$w\varphi \geqslant 0, \quad \varphi \neq 0; \qquad w\varphi < 0, \quad \psi_1 > 0$$

G^2 spherical geometry group $\dim H = 6$;

$$w\varphi < 0, \qquad \psi_1 < 0$$

G^3 Lorentz group (isomorphic to the group of plane transformations, time taken as the geometrical coordinate) $\dim H = 5; \quad m = 2$

$$\varphi = 0, \quad \psi_i \geqslant 0 \quad (1 = 1, 2, 3), \qquad \psi_1^2 + \psi_2^2 + \psi_3^2 \neq 0$$

G^4 Euclidean group $\dim H = 5$;

$$\varphi = 0, \quad \psi_i \leqslant 0 \quad (i = 1, 2, 3), \qquad \psi_1^2 + \psi_2^2 + \psi_3^2 \neq 0$$

G^5 Galilean group $\dim H = 3; m = 6$

$$\varphi = 0, \quad \psi_i = 0, \quad u^2 + v^2 + w^2 \neq 0$$

G^6 commutative group $\dim H = 0; m = 7$

$$\varphi = 0, \quad \psi_i = 0, \quad u^2 + v^2 + w^2 = 0$$

Groups G belonging to the infinite (continuous) series $(f_1^2 + + f_2^2 + f_3^2 > 0)$.

G^{c_0} $\dim H = 5, \ m = 2.5$

$$\chi_1 = \chi_2 = \chi_3 = 0 \quad (\varphi = 0), \qquad \Omega = c_0 \neq 0, \ \infty$$

G^7 $\dim H = 5; \ m = 4$

$$\chi_1 = \chi_2 = \chi_3 = 0 \quad (\varphi = 0), \qquad \Omega = 0, \qquad u^2 + v^2 + w^2 \neq 0$$

G^8 $\dim H = 3; \ m = 3$

$$\chi_1 = \chi_2 = \chi_3 = 0 \quad (\varphi = 0) \qquad \Omega = 0, \qquad u^2 + v^2 + w^2 = 0$$

The orbits of T lying on the manifold Γ_1 correspond to the groups of the finite series and the orbits of T lying on Γ_2, to the groups of the infinite series. Symbol G^{c_0} denotes the group continuum: each pair of distinct real values of c_0 has a corresponding pair of nonisomorphic groups of G. Passage to the limit $c_0 \to \infty$ formally yields the Lorentz group. For the groups of the type G^{c_0}, the following commutative relations are pointed out by G. Scheffers:

$$(X_1, X_2) = 0, \quad (X_1, X_3) = X_1, \quad (X_2, X_3) = cX_2 \ (c \neq 0, 1) \tag{2.4}$$

which in turn yield

$$\Omega = c_0 = \left(\frac{1-c}{1+c}\right)^2$$

Thus, $c = -1$ corresponds to the Lorentz group. The same group corresonds to both values, $c' = c$ and $c'' = 1 / c$, therefore all algebras defined by (2.4) are obtained at $-1 \leqslant c \leqslant 1$. However, in this case the constant c_0 assumes positive values only, and to obtain $c_0 < 0$ one would have to set $c = e^{i\varphi}$ $(0 < \varphi < \pi)$.

Figure 1 shows the adjacency diagram for the groups G. Each column contains the groups for which the orbits H belong to the same manifold Γ and each row contains the groups whose corresponding orbits are of the same dimension. The arrows indicate the adjacencies. The diagram is transitive: if G' is adjacent to G'' and G'' to G''', then G' is adjacent to G'''.

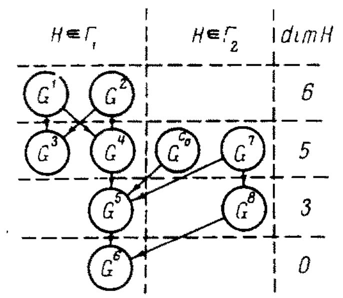

Fig. 1

All groups except G^1 and G^2 have a solution. The groups $G^1, \ldots, G^5$ of the finite series describe five out of nine Cayley-Klein geometries ([8], p. 210). The diagram shows that the Euclidean group is adjacent to both, the Lobachevskii and the local spherical geometry group, the fact established by Riemann in geometrical terms ([8], pp. 155, 156). Thus the geometries listed above are the only geometries refining the plane Euclidean geometry. All these groups can be realized as the groups of motion of a two-dimensional Riemannian space of constant curvature ([9], p. 273). The Lorentz and Euclidean groups are directly adjacent to the Galilean group. This is another well know fact established by Poincaré and Einstein in the course of investigation of the fundamental properties of the space-time. In addition the infinite series of groups G^{c_0} is also adjacent to the Galilean group. Since, as we already said, the Lorentz group is formally related to their number, a question arises, why out of all groups G^{c_0} it is the Lorentz group that makes the properties of the Galilean space-time more precise.

As we know from [6], (p. 396), the Lie group of transformation can be reconstructed in terms of its structural constants only with the accuracy to within the similarity (in other words, with the accuracy allowing the choice of a coordinate system arithmetizing the transformed space). For this reason the knowledge of the groups adjacent to the given group does not present us with all the possible improvements in the theory described by this group. We follow with some examples of algebras AG for the groups G of transformations.

Group G^1. In the polar coordinates u, v we have

$$X_1 = \frac{\partial}{\partial v}; \qquad X_2 = \sin v \frac{\partial}{\partial u} + \frac{G'(u)}{2G(u)} \cos v \frac{\partial}{\partial v}$$

$$X_3 = \cos v \frac{\partial}{\partial u} - \frac{G'(u)}{2G(u)} \sin v \frac{\partial}{\partial v}, \qquad G(u) = \left(\frac{\operatorname{sh} \sigma u}{\sigma}\right)^2$$

The commutation relations are

$$(X_1, X_2) = X_3, \ (X_1, X_3) = -X_2, \ (X_2, X_3) = -\sigma^2 X_1$$

and the transformations of the group preserve the metric ([10] p. 89)

$$ds^2 = du^2 + G(u)\, dv^2$$

Group G^2 is obtained from G^1 by means of the transformation

$$\sigma \to i\sigma$$

Group G^3 (x is the geometrical coordinate and t is time)

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial t}, \qquad X_3 = t\frac{\partial}{\partial x} + \mu^2 x \frac{\partial}{\partial t}$$

The commutation relations are

$$(X_1, X_2) = 0, \qquad (X_1, X_3) = \mu^2 X_2, \qquad (X_2, X_3) = X_1$$

and its transformations preserve the metric

$$ds^2 = -\mu^2 dx^2 + dt^2$$

Group G^4 (x, y are Cartesian coordinates)

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial y}, \qquad X_3 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$$

The commutation relations are

$$(X_1, X_2) = 0, \qquad (X_1, X_3) = -X_2, \qquad (X_2, X_3) = X_1$$

and its transformations preserve the metric

$$ds^2 = dx^2 + dy^2$$

(G^4 can also be obtained from G^3 by means of transformation $\mu \to i\mu$).

Group G^5 (x is the geometrical coordinate and t is time)

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial t}, \qquad X_3 = t\frac{\partial}{\partial x}$$

The commutation relations are

$$(X_1, X_2) = 0, \qquad (X_1, X_3) = 0, \qquad (X_2, X_3) = X_1$$

and no metric preserved by its transformations exists.

Groups G^{c_0}

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial t}, \qquad X_3 = (t + c_0 x)\frac{\partial}{\partial t} + (t + x)\frac{\partial}{\partial x}$$

The commutation relations are

$$(X_1, X_2) = 0 \qquad (X_1, X_3) = X_1 + c_0 X_2, \qquad (X_2, X_3) = X_1 + X_2$$

The metric preserved under the transformations of the groups, if it exists, can be found by integrating the Killing equations ([9], p. 251).

3. Invariants and invariant manifolds of the group T. Let us obtain the components $\zeta_{\beta ij}^{\alpha\delta}$ of the infinitesimal operators

$$Y_\beta^{\ \alpha} = \zeta_{\beta ij}^{\alpha\delta} \frac{\partial}{\partial c_{ij}^{\ \delta}} \ (i < j)$$

of the algebra AT of the group T.

Differentiating (2.1) with respect to $\alpha_\alpha^{\beta'}$ we obtain

$$\frac{\partial c_{ij}^{\ \delta}}{\partial \alpha_\alpha^\beta} = c_{\beta k_2}^{\ \gamma} \delta_i^\alpha \alpha_j^{k_2} \alpha_\gamma^{*\delta} + c_{k_1\beta}^{\ \gamma} \alpha_i^{k_1} \delta_j^\alpha \alpha_\gamma^{*\delta} + \alpha_i^{k_1} \frac{\partial \alpha_\gamma^{*\delta}}{\partial \alpha_\alpha^\beta} c_{k_1 k_2}^{\ \gamma} \alpha_j^{k_2}$$

$$\frac{\partial \alpha_\varepsilon^{*\delta}}{\partial \alpha_\alpha^\beta} = -\frac{\partial \alpha_\nu^\alpha}{\partial \alpha_\alpha^\beta} \alpha_\gamma^{*\delta} \alpha_\varepsilon^{*\nu} = -\delta_\beta^\gamma \delta_\alpha^\nu \alpha_\gamma^{*\delta} \alpha_\varepsilon^{*} = -\alpha_\beta^{*\delta} \alpha_\varepsilon^{*\alpha}$$

We take as $\zeta_{\beta ij}^{\alpha\delta}$ the derivatives $\partial c_{ij}^{\ \delta'} / \partial \alpha_\alpha^\beta$ accompanying the values $\alpha_\alpha^\beta = \alpha_\alpha^{*\beta} = \delta_\alpha^\beta$ corresponding to the identity element of the group $GL(n, R)$

$$\zeta_{\beta ij}^{\alpha\delta} = \delta_i^\alpha c_{\beta j}^{\ \delta} + \delta_j^\alpha c_{\beta i}^{\ \delta} - \delta_\beta^\delta c_{ij}^{\ \alpha} \tag{3.1}$$

The commutation relations in the resulting basis of the algebra AT have the form

$$(Y_\beta^\alpha, Y_{\beta'}^{\alpha'}) = \delta_{\beta'}^{\alpha} Y_\beta^{\alpha'} - \delta_\beta^{\alpha'} Y_{\beta'}^{\alpha}.$$

Using the notation (2.3) we can write the Jacobi equations (1.1) as

$$\chi_1 = \chi_2 = \chi_3 = 0 \tag{3.2}$$

From this it follows that either

$$f_1^2 + f_2^2 + f_3^2 = 0 \tag{3.3}$$

or $f_1^2 + f_2^2 + f_3^2 > 0$, and then

$$2\varphi = \begin{vmatrix} 2u & -\theta_3 & \theta_2 \\ \theta_3 & -2w & -\theta_1 \\ \theta_2 & \theta_1 & -2v \end{vmatrix} = 2(\theta_1\theta_2\theta_3 + 4uvw + u\theta_1^2 + w\vartheta_2^2 - v\theta_3^2) = 0 \tag{3.4}$$

It can be shown that (3.4) is equivalent to the Cartan condition of solvability of the groups G.

In the variables u, v, w, f_i, θ_i $(i = 1, 2, 3)$ the matrix of the infinitesimal operators of the algebra AT assumes the form

	ζ_u	ζ_v	ζ_w	ζ_{θ_1}	ζ_{θ_2}	ζ_{θ_3}	ζ_{f_1}	ζ_{f_2}	ζ_{f_3}	
Y_1^1	$-u$	v	w	θ_1	0	0	f_1	0	0	
Y_1^2	θ_2	0	0	0	$-2v$	$-\theta_1$	0	0	$-f_1$	
Y_1^3	$-\theta_3$	0	0	0	θ_1	$-2w$	0	$-f_1$	0	
Y_2^1	0	$-\theta_2$	0	$-\theta_3$	$2u$	0	$-f_3$	0	0	
Y_2^2	u	$-v$	w	0	0	θ_3	0	0	f_3	(3.5)
Y_2^3	0	θ_1	0	$-2w$	θ_3	0	0	f_3	0	
Y_3^1	0	0	$-\theta_3$	θ_2	0	$-2u$	$-f_2$	0	0	
Y_3^2	0	0	$-\theta_1$	$2v$	0	θ_2	0	0	f_2	
Y_3^3	u	v	$-w$	0	θ_2	0	0	f_2	0	

It can easily be shown that (3.3) and (3.2) together with (3.4) define the invariant manifolds of the group T. The set of all invariant manifolds of the group induced by T on the first manifold Γ_1 forms the sequence

$$\Gamma_1 \supset \Gamma_1^1 \supset \Gamma_1^2 \supset \Gamma_1^3$$

The group induced by T on the second manifold Γ_2 is intransitive. Its transformations leave a continuous set of surfaces $\Gamma_2^{c_0} \subset \Gamma_2$ corresponding to the invariants of T unaffected, with varying real values of c_0 filling the whole manifold Γ_2. The group T exists and is locally transitive on the surfaces $\Gamma_2^{c_0}$ at all general points, admitting the invariant manifolds Γ_2^1 and $\Gamma_2^2 \subset \Gamma_2^1$ at $c_0 = 0$.

Using the standard methods of computing the invariants and invariant manifolds ([9], pp. 84–89), we obtain

$$\Gamma_1 = \{f_1 = f_2 = f_3 = 0\}; \qquad \Gamma_1^1 = \{f_1 = f_2 = f_3 = 0, \quad \varphi = 0\}$$

$$\Gamma_1^2 = \{f_1 = f_2 = f_3 = 0, \ \varphi = 0, \ \psi_i = 0\} \quad (i=1,\dots,6)$$

$$\Gamma_1^3 = \{f_1 = f_2 = f_3 = 0, \ \varphi = 0, \ \psi_i = 0, \ u = v = w = 0\}$$

$$\Gamma_2 = \{\chi_1 = \chi_2 = \chi_3 = 0\}, \qquad \Gamma_2^{c_0} = \{\chi_1 = \chi_2 = \chi_3 = 0, \varphi = 0, \Omega = c_0\}$$

$$\Gamma_2^1 \ (\Gamma_2^{c_0} \ \text{at} \ c_0 = 0) = \{\chi_1 = \chi_2 = \chi_3 = 0, \ \varphi = 0, \ \Omega = 0\} \ (c_0 \neq 0)$$

$$\Gamma_2^2 = \{\chi_1 = \chi_2 = \chi_3 = 0, \ \varphi = 0, \ \Omega = 0, \ u = v = w = 0\}$$

We note that the tensorial character of the structural constants and the general theory of algebraic invariants [11] together imply that tensor contraction and alternation may be employed to obtain the invariants and invariant manifolds of the group T.

4. Orbits of the group T. Here we shall use several simple assertions, the first of which is obvious.

4.1°. Let the set A belonging to the space R be defined by the following system of simultaneous algebraic inequalities

$$\varphi_i(y_1, \dots, y_m; x_1, \dots, x_n) \geqslant 0 \quad (i = 1, \dots, m), \ \sum_{j=1}^{k} \omega_j^2(x_1, \dots, x_n) \neq 0$$

the left-hand sides of which are such that (a) the nonempty set B defined by the conditions $\omega_j > 0$ ($j = 1, \dots, k$) is connected and (b) the system of equations $\varphi_i = 0$ ($i = 1, \dots, m$) has a unique solution in the variables $y_1, \dots, y_m$ at any point of B. Then the set $A \cap B$ defined by the inequalities $\varphi_1 \geqslant 0, \dots, \varphi_m \geqslant 0, \ \omega_1 > 0, \dots, \omega_k > 0$ is connected.

4.2°. Let C and D be sets belonging to R and considered as subspaces and let one of the mappings

$$g_1: c_{12}^{1'} = -c_{12}^1, \ c_{23}^{3'} = -c_{23}^3, \ c_{13}^{1'} = c_{13}^1, \ c_{23}^{2'} = c_{23}^2, \ c_{12}^{2'} = c_{12}^2, \ c_{13}^{3'} = c_{13}^3$$

$$g_2: c_{12}^{1'} = c_{12}^1, \ c_{23}^{3'} = c_{23}^3, \ c_{13}^{1'} = -c_{13}^1, \ c_{23}^{2'} = -c_{23}^2, \ c_{12}^{2'} = c_{12}^2, \ c_{13}^{3'} = c_{13}^3$$

$$g_3: c_{12}^{1'} = c_{12}^1, \ c_{23}^{3'} = c_{23}^3, \ c_{13}^{1'} = c_{13}^1, \ c_{23}^{2'} = c_{23}^2, \ c_{12}^{2'} = -c_{12}^2, \ c_{13}^{3'} = -c_{13}^3$$

(with $u' = -u$, $v' = -v$ and $w' = -w$ in all three mappings) be a homeomorphism of the set C on the set D. Then C and D both correspond to the same local group G. In this case we shall speak of the sets C and D as connected by a reflection and assume them to be not essentially different.

Proof. The commutation relations

$$(X_i, X_j) = c_{ij}^k X_k \qquad (i, j, k = 1, 2, 3)$$

of the algebra AG show at once that the mappings g_1, g_2 and g_3 are generated by the reflections $X_2' = -X_2$, $X_3' = -X_3$, $X_1' = -X_1$ respectively.

4.3°. When $f_1^2 + f_2^2 + f_3^2 = 0$ we have, by (2.3), the following identities:

$$\varphi c_{12}^1 = \psi_4\psi_6 + \psi_3\psi_5, \quad \varphi c_{13}^1 = \psi_2\psi_4 + \psi_5\psi_5, \quad \varphi c_{12}^2 = \psi_4\psi_5 + \psi_1\psi_6$$

$$\varphi u = \psi_6^2 - \psi_2\psi_3, \qquad \varphi v = \psi_1\psi_2 - \psi_5^2, \qquad \varphi w = \psi_4^2 - \psi_1\psi_3$$

$$\varphi^2 = \psi_1\psi_6^2 + \psi_2\psi_4^2 + \psi_3\psi_5^2 + 2\psi_4\psi_5\psi_6 - \psi_1\psi_2\psi_3 \qquad \text{(cont.)}$$

Their validity can be checked by direct verification.

The orbits of T are found, one after the other, for all sets $\Gamma^{\alpha-1}\setminus\Gamma^\alpha$ (see Sect. 2).

Set $\Gamma_1\setminus\Gamma_1^1$. If the sign of the function φ is given, then by 4.3°, the variables c_{12}^1, c_{13}^1, c_{12}^2, $c_{23}^1 = u$, $c_{13}^2 = v$ and $c_{12}^3 = w$ are expressed uniquely in terms of ψ_i as parameters. The latter quantities are constrained here by the relation

$$\psi_1\psi_6^2 + \psi_2\psi_4^2 + \psi_3\psi_5^2 + 2\psi_4\psi_5\psi_6 - \psi_1\psi_2\psi_3 > 0$$

Let us denote

$$p = -\frac{q}{\psi_4^2 - \psi_1\psi_3} = -\frac{\psi_1\psi_6^2 + \psi_3\psi_5^2 + 2\psi_4\psi_5\psi_6}{\psi_4^2 - \psi_1\psi_3}$$

By 4.1° and 4.2° each of the sets given below consists of two connected parts joined by a reflection

$$\varphi \neq 0, \quad \psi_4^2 - \psi_1\psi_3 > 0, \qquad (\text{hence } \psi_2 > p) \tag{4.1}$$

$$\varphi \neq 0, \quad \psi_4^2 - \psi_1\psi_3 < 0,\ \psi_1 > 0 \qquad (\text{hence } \psi_3 > 0,\ \psi_2 < p) \tag{4.2}$$

$$\varphi \neq 0, \quad \psi_4^2 - \psi_1\psi_3 < 0, \quad \psi_1 < 0 \quad (\text{hence } \psi_3 < 0,\ \psi_2 < p) \tag{4.3}$$

When $\psi_4^2 - \psi_1\psi_3 < 0$ and $\psi_1 > 0$ the function q taken as a homogeneous quadratic form in ψ_5 and ψ_6 is positive definite. Since $q \to \infty$ when $\psi_4^2 - \psi_1\psi_3 \to -0$ (otherwise $\varphi \to 0$), $p \to \infty$. Since $\psi_2 < p$, a continuous passage exists from the set (4.2) to either of the two connected parts of the set

$$\varphi \neq 0, \quad \psi_4^2 - \psi_1\psi_3 = 0, \quad \psi_1 > 0 \tag{4.4}$$

On the other hand, when $\psi_4^2 - \psi_1\psi_3 > 0$, the form q has an alternating sign and the variables can be chosen in such a manner that $q > 0$ without violating the remaining conditions. Then $p \to -\infty$ as $\psi_4^2 - \psi_1\psi_3 \to 0$ and the fact that $\psi_2 > p$ implies that a continuous passage from the set (4.1) to the set (4.4) is possible.

Thus the union M_1 of the sets (4.1), (4.2) and (4.4) belongs to the same orbit. We can define this orbit, in accordance with 4.3°, as the union of two nonintersecting sets defined by the inequalities

$$M_1 = \{w\varphi \geqslant 0,\ \varphi \neq 0;\ w\varphi < 0,\ \psi_1 > 0\}$$

Denoting by M_2 the set defined by (4.3) (according to 4.3° it can also be defined by the inequalities $w\varphi < 0$ and $\psi_1 < 0$) we obviously have $M_1 \cup M_2 = \Gamma_1\setminus\Gamma_1^1$. We shall show that each of the sets M_1 and M_2 is an orbit of the manifold Γ_1. For this it is sufficient to establish that the boundary $\overline{M}_2 \setminus M_2$ of the set M_2 cannot be reached from within this set. Ideed, when $\psi_4^2 - \psi_1\psi_3 < 0$ and $\psi_1 < 0$, the form q is negative definite so that $p \to -\infty$ as $\psi_4^2 - \psi_1\psi_3 \to 0$. Then the condition $\psi_2 < p$ indicates a violation of the connectivity during the passage from the set M_2 to its closure $\overline{M}_2$.

Set $\Gamma_1^1 \setminus \Gamma_1^2$. This set is defined by the conditions $f_1^2 + f_2^2 + f_3^2 = 0$, $\varphi = 0$ and $\psi_1^2 + \psi_2^2 + \psi_3^2 \neq 0$ (if $\psi_1^2 + \psi_2^2 + \psi_3^2 = 0$ then 4.3° yield the equations $\psi_i = 0$ ($i = 1, \ldots, 6$) defining the set Γ_1^2). From 4.1° it follows that the subsets of $\Gamma_1^1\setminus\Gamma_1^2$ given below are connected

$$K_1 = \{\varphi = 0,\ \psi_1 > 0\}, \quad L_1 = \{\varphi = 0,\ \psi_1 < 0,\ w > 0\}$$
$$N_1 = \{\varphi = 0,\ \psi_1 < 0,\ w < 0\}$$

$$K_2 = \{\varphi = 0,\ \psi_2 > 0\},\quad L_2 = \{\varphi = 0,\ \psi_2 < 0,\ v > 0\} \qquad \text{(cont.)}$$
$$N_2 = \{\varphi = 0,\ \psi_2 < 0,\ v < 0\}$$
$$K_3 = \{\varphi = 0,\ \psi_3 > 0\},\quad L_3 = \{\varphi = 0,\ \psi_3 < 0,\ w > 0\}$$
$$N_3 = \{\varphi = 0,\ \psi_3 < 0,\ w < 0\}$$

According to 4.2° the sets L_i and N_k are connected by a reflection for $i = k$. By 4.3° the functions ψ_1, ψ_2, ψ_3 cannot have different signs at the same time, at $\varphi = 0$. The sets K_i and $L_i \cup N_i$ must therefore be determined by the inequalities

$$K_1 = \{\varphi = 0,\ \psi_1 > 0,\ \psi_2 \geqslant 0,\ \psi_3 \geqslant 0\},\quad L_1 \cup N_1 = \{\varphi = 0,\ \psi_1 < 0\ \psi_2 \leqslant 0,\ \varphi_3 \leqslant 0\}$$
$$K_2 = \{\varphi = 0,\ \psi_1 \geqslant 0,\ \psi_2 > 0,\ \psi_3 \geqslant 0\},\quad L_2 \cup N_2 = \{\varphi = 0,\ \psi_1 \leqslant 0\ \psi_2 < 0,\ \psi_3 \leqslant 0\}$$
$$K_3 = \{\varphi = 0,\ \psi_1 \geqslant 0,\ \psi_2 \geqslant 0,\ \psi_3 > 0\},\quad L_3 \cup N_3 = \{\varphi = 0,\ \psi_1 \leqslant 0\ \psi_2 \leqslant 0,\ \psi_3 < 0\}$$

From this it follows that $K_1 \cap K_2 \cap K_3 \neq \Lambda$ so that the set $K_1 \cup K_2 \cup K_3$ is connected and defined by the conditions (4.5)

$$f_1^2 + f_2^2 + f_3^2 = 0,\quad \varphi = 0,\ \psi_1 \geqslant 0,\ \psi_2 \geqslant 0,\ \psi_3 \geqslant 0,\ \psi_1^2 + \psi_2^2 + \psi_3^2 \neq 0$$

The intersection of all sets $L_i \cup N_i$ is also nonempty. This implies that their union is connected and defined by the conditions (4.6)

$$f_1^2 + f_2^2 + f_3^2 = 0,\ \varphi = 0,\ \psi_1 \leqslant 0,\ \psi_2 \leqslant 0,\ \psi_3 \leqslant 0,\ \psi_1^2 + \psi_2^2 + \psi_3^2 \neq 0$$

Any continuous curve connecting the points of the sets (4.5) and (4.6) obviously intersects the set Γ_1^2. This means that each of the sets (4.5) and (4.6) forms an orbit lying on Γ_1.

Set $\Gamma_1^2 \setminus \Gamma_1^3$. We shall show that this set decomposes into two connectivity components joined by a reflection, and forms therefore a single orbit of the set Γ_1

$$f_1^2 + f_2^2 + f_3^2 = 0,\quad \varphi = 0,\ \psi_1^2 + \psi_2^2 + \psi_3^2 = 0,\ u^2 + v^2 + w^2 \neq 0$$

Indeed, according to 4.1° each of the sets

$$f_i = \psi_i = \varphi = 0,\quad u > 0;\qquad f_i = \psi_i = \varphi = 0,\quad u < 0$$
$$f_i = \psi_i = \varphi = 0,\quad v < 0;\qquad f_i = \psi_i = \varphi = 0,\quad v > 0$$
$$f_i = \psi_i = \varphi = 0,\quad w > 0;\qquad f_i = \psi_i = \varphi = 0,\quad w < 0$$

is connected. When the conditions $\psi_1 = \psi_2 = \psi_3 = 0$ hold, only the following combinations of signs are possible:

$$u \geqslant 0,\quad v \leqslant 0,\quad w \geqslant 0,\quad u^2 + v^2 + w^2 \neq 0$$

or

$$u \leqslant 0,\quad v \geqslant 0,\quad w \leqslant 0,\quad u^2 + v^2 + w^2 \neq 0$$

This implies that the set $\Gamma_1^2 \setminus \Gamma_1^3$ decomposes into two connected sets

$$f_i = \psi_i = \varphi = 0,\quad u \geqslant 0,\quad v \leqslant 0,\quad w \geqslant 0,\quad u^2 + v^2 + w^2 \neq 0$$
$$f_i = \psi_i = \varphi = 0,\quad u \leqslant 0,\quad v \geqslant 0,\quad w \leqslant 0,\quad u^2 + v^2 + w^2 \neq 0$$

joined by a reflection.

Set Γ_1^3 consists of a single point, it forms a unique orbit.

Set $\Gamma_2^{c_0}$ $(c_0 \neq 0)$. We shall distinguish two cases, $c_0 < 0$ and $c_0 > 0$.

(1) For $c_0 < 0$ we have $w \neq 0$ and the equations

$$\varphi = 0, \qquad \theta_1^2 + 4vw = c_0 f_1^2$$

have a unique solution in u and v

$$u = -\frac{-v\theta_3^2 + w\theta_2^2 + \theta_1\theta_2\theta_3}{c_0} f_1^2, \qquad v = \frac{c_0 f_1^2 - \theta_1^2}{w}$$

According to 4.1° and 4.2° we have two connected sets defined on the set $\Gamma_2^{c_0}$ by the conditions $w > 0$ and $w < 0$ respectively and joined by a reflection.

(2) For $c_0 > 0$ we have $\theta_1^2 + 4vw > 0$.

Since $f_1\sqrt{c_0} = \pm\sqrt{\theta_1^2 + 4vw}$ and the set $\theta_1^2 + 4vw > 0$ is connected, the set $\Gamma_2^{c_0}$ decomposes for $c_0 > 0$ into two connected subsets (corresponding to the conditions $f_1 > 0$ and $f_1 < 0$) joined by a reflection. Thus for any real value of c_0 the set $\Gamma_2^{c_0}$ $(c_0 \neq 0)$ defines an orbit belonging to Γ_2.

Set $\Gamma_2^1 \setminus \Gamma_2^2$ is defined by the conditions

$$\chi_1 = \chi_2 = \chi_3 = 0, \quad \varphi = 0, \ \Omega = 0, \quad u^2 + v^2 + w^2 \neq 0, \ \psi_i = 0$$

and forms a unique orbit of Γ_2. This can be proved in the manner analogous to that used for $\Gamma_1^2 \setminus \Gamma_1^3$.

Set Γ_2^2 defined by the conditions $u = v = w = \theta_1 = \theta_2 = \theta_3 = 0$ is obviously connected, hence it defines a unique orbit of Γ_2.

5. Adjacency diagram. Let $P^\alpha = \Gamma_j^{\alpha-1} \setminus \Gamma_j^\alpha$ and let H be the connectivity component of the subspace P^α in R. Then H is closed on P^α. We have

$$H = \overline{H}_{P^\alpha} = \overline{H} \cap (\Gamma_j^{\alpha-1} \setminus \Gamma_j^\alpha) = (\overline{H} \setminus \Gamma_j^\alpha) \cap \Gamma_j^{\alpha-1} = \overline{H} \setminus \Gamma_j^\alpha$$

From this it follows that $\overline{H} \setminus H \subset \Gamma^\alpha$. Consequently the group G corresponding to the orbit $H \subset P^\alpha \subset \Gamma_j^{\alpha-1}$ is adjacent to those groups, whose orbits belong to the invariant manifold Γ_j^α. This fact is used to determine the adjacency in the groups $G^1, \ldots, G^6$ and partly in the groups of the continuous series. In addition, the orbits corresponding to the groups of the continuous series include the segments of the boundary defined in R by the intersection

$$\Gamma_1 \cap \overline{\Gamma}_2 = \{f_1^2 + f_2^2 + f_3^2 = 0, \ \varphi = 0\}$$

and this forms the condition of adjacency of the groups of the continuous series to the groups G^5 and G^6 of the finite series.

In Sect. 4 all orbits are defined in terms of the algebraic inequalities. Let us e. g. verify the closure of the orbit $M_2 \subset \Gamma_1 \setminus \Gamma_1^1$ on the subspace $\Gamma_1 \setminus \Gamma_1^1$ using its definition

$$(4.3) \qquad \psi_4^2 - \psi_1\psi_3 < 0, \ \psi_1 < 0, \ \psi_3 < 0, \ \varphi^2 > 0$$

(the closure of the remaining orbits is obvious). For this it is sufficient to show that none of these inequalities can become an equality while the rest of them remain inequalities. Suppose that $\psi_1 = 0$. Then $\psi_4^2 - \psi_1\psi_3 \geqslant 0$. If $\psi_4^2 - \psi_1\psi_3 = 0$ and $\varphi \neq 0$, then by 4.3° $w = 0$ and $\psi_1 \geqslant 0$. Consequently the equalities $\psi_1 = 0$ and $\psi_4^2 - \psi_1\psi_3 = 0$ must hold simultaneously. But then the relation $\varphi^2 = \psi_3\psi_5^2 > 0$ implies that $\psi_3 > 0$. Consequently the equalities $\psi_1 = \psi_3 = \psi_4^2 - \psi_1\psi_3 = 0$ must hold simulteneously and this implies that $\varphi = 0$, the latter defining the invariant manifold Γ_1^1.

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ON CERTAIN DIMENSION PROPERTIES OF A CONTROL STABILIZING A MECHANICAL SYSTEM

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The problem of determining the smallest number of controls stabilizing the equilibrium position of a mechanical system is investigated. Necessary and sufficient conditions are established under which stabilization of the equilibrium position is possible with a control of minimal dimension, and this dimension is determined. The influence of gyroscopic and dissipative forces on the dimension of the stabilizing control is studied completely for a linear approximation of the system being considered. Necessary conditions are found under which stabilization is possible by forces which depend only on the velocity.

1. We consider a controlled conservative mechanical system with n degrees of freedom, whose motion is described by the Lagrange equations

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} + \frac{\partial \Pi}{\partial q_i} = Q_i(u_1, \ldots, u_r), \quad Q_i(0, \ldots, 0) = 0 \quad (i = 1, \ldots, n) \qquad (1.1)$$